

COLORIMETRY

In photometry a lumen is a lumen, no matter what wavelength or wavelengths of light are involved. But it is that combination of wavelengths that produces the sensation of color, one of the most significant properties of light. While photometry deals only with the amount of light, colorimetry deals with the quantitative specification of the color of light. Colorimetric principles have applications in photography, television, printing, etc.

COLOR MATCHING

Colorimetry is based on the fact that most colors may be matched by superimposing light from three sources, typically a red, green, and blue source. These are called the primary colors or just primaries. Extending the notion of color matching with a little mathematical chicanery it is, in fact, possible to match *any* color of light with three primaries.

Suppose, for example, we have projectors which produce spectrally pure 700 nm red light, 546 nm green light, and 436 nm blue light. If we project light so that the illuminance of the sources on a white receiving screen are 100 lm/sq m of red light, 459 lm/sq m of green light, and 6 lm/sq m of blue light, as in the diagram below, we'll find that the area where all three sources overlap has an illuminance of 565 lm/sq m of white light.

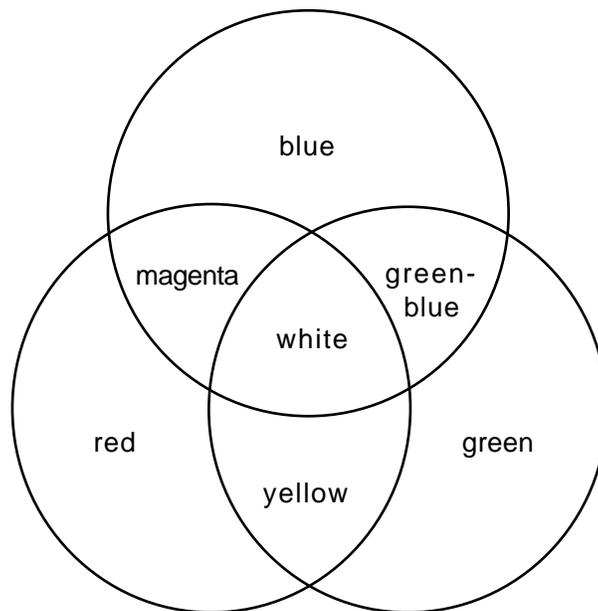


Figure 1. Overlapping red, green, and blue lights produce white light.

If we were to increase the illuminance of 700 nm red light 100 lm/sq m to 200 lm sq/m, the superimposition of the three lights on a white screen would be pink or a desaturated red, that is, a red with white mixed in, pink, if you like. Its illuminance would be 665 lm/sq m. The hue or dominant wavelength of the combination would be that of 700 nm red light. Pure spectral colors contain no white component and are said to be saturated. Spectral colors can be produced by a monochromator which uses a prism or grating.

As long as the proportions of the color components are kept the same, the hue and saturation of the combination remains the same, only the illuminance changing. Overlapping 200 lm/sq m of red light, 918 lm/sq m of green light, and 12 lm/sq m of blue light would produce the same white light as before, but with 1130 lm/sq m illuminance, twice as much as before. Likewise, 100 lm/sq m of red light, 230 lm/sq m of green light, and 3 lm/sq m of blue light would produce the same pink light as in the example above, but with the illuminance halved to 333 lm/sq m.

The experiment underlying all of colorimetry is that of color matching. A typical field of view in a color matching experiment is shown in figure 2. The task of the experimental subject is to adjust the luminances of red, green, and blue lights on the left side of the field to match the colored field at the right.

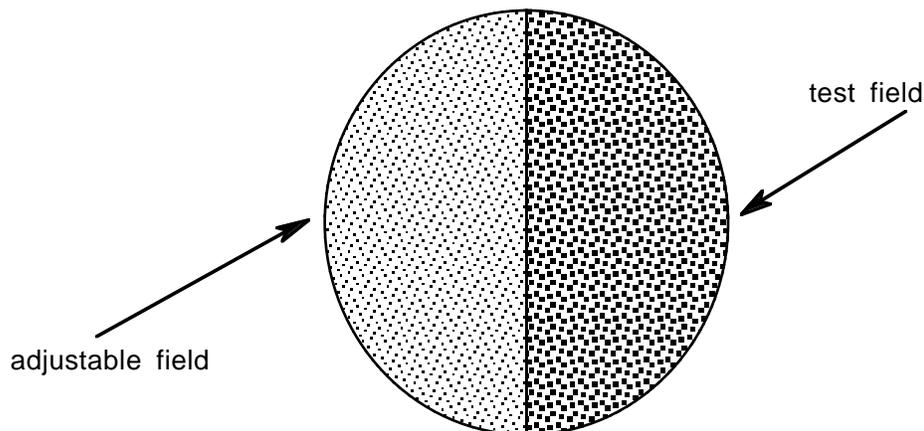


Figure 2. In a colorimetry experiment a subject would adjust the luminances of red, green, and blue on the left hand side of the field to match a colored field on the right. The field typically subtends 2° of arc at the eye.

When this experiment is performed, subjects find that they can always

match the luminance and hue of the test stimulus, but often they cannot match its saturation. In such cases the hue has been matched using two of the primaries. To complete the match, the experimenter allows the subject to add the third primary to the test field until a match has been obtained. This is equivalent to adding a *negative* luminance to the adjustable field. If, for example, a match has been obtained using 100 lx of red light, 50 lx of green light, with 30 lx of blue light added to the test field, that is mathematically equivalent to matching the test field with 100 lx of red light, 50 lx of green light, and -30 lx of blue light.

The negative luminance has a purely mathematical meaning. On paper we can add together primaries with combinations of positive and negative luminances mathematically which correspond to no real physical color. These are the so-called imaginary colors, and no one has or ever will actually see them. Unfortunately, these imaginary colors have a perversely important role in the usual formulation of colorimetry, as we shall see.

In colorimetry it is not very convenient to quantify stimuli in the usual photometric units based on luminous flux. Instead the luminance, L , of a stimulus is given in terms of tristimulus values R , G , and B by the equation

$$L(\text{lx}) = R + 4.5907 G + 0.0601 B$$

when using the 700 nm, 546 nm, and 436 nm primaries we discussed above. This serves to quantize the units of the different primaries. This particular quantization gives white light whenever $R = G = B$.

We can think of tristimulus values as a three dimensional vector (R, G, B) . In this notation the white of our first example would have components (100,100,100). The pink color in the example would be (200,100,100). A spectral 700 nm red of 300 lx luminance would be represented by (300,0,0).

If we were to double the luminance of a stimulus with tristimulus values (R, G, B) , the resultant stimulus would have tristimulus values $2(R, G, B) = (2R, 2G, 2B)$. If we were to combine stimuli with tristimulus values (R_1, G_1, B_1) and (R_2, G_2, B_2) , the combination would have tristimulus values $(R, G, B) = (R_1, G_1, B_1) + (R_2, G_2, B_2) = (R_1 + R_2, G_1 + G_2, B_1 + B_2)$. Thus the tristimulus value vector obeys the usual laws of vector addition.

The color matching functions or distribution coefficients r_λ , g_λ , and b_λ are special tristimulus values which apply to a spectral source of wave length λ

and unit energy.¹ The tristimulus values for a spectral source of power P_λ is $(R, G, B) = P_\lambda(r_\lambda, g_\lambda, b_\lambda)$. The distribution functions determined from color matching experiments are plotted in figure 3. Note how the values of all the curves but that associated with the relevant primary vanish at 700 nm, 546 nm, and 436 nm. And also note the negative values for many of the distribution coefficients.

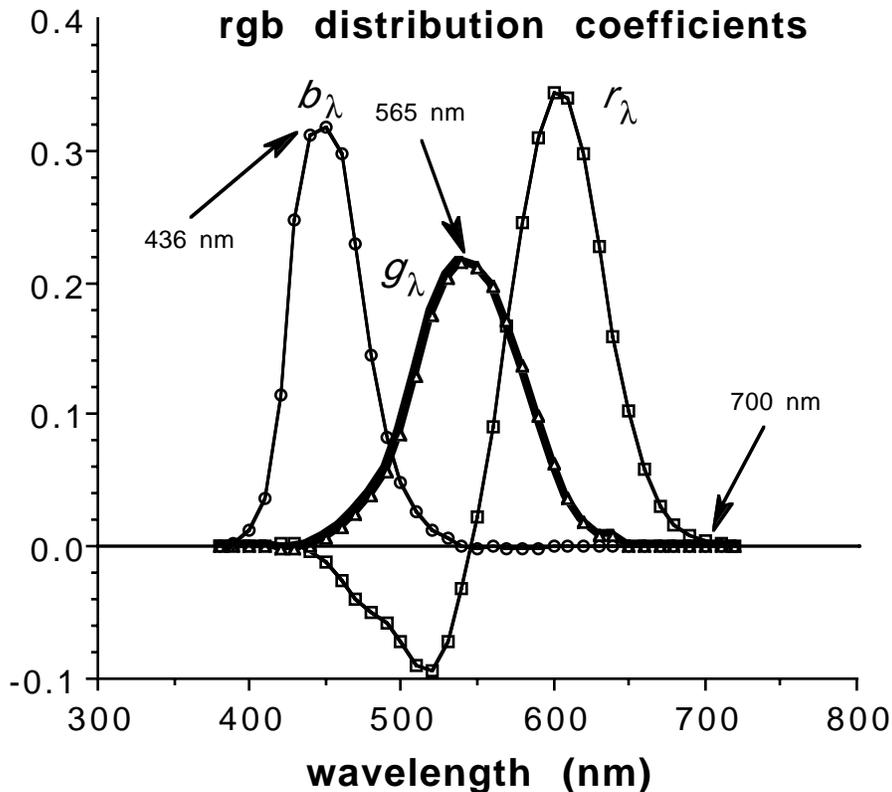


Figure 3. The distribution coefficients in the rgb color specification system.

Color specification is usually concerned only with the relative proportions of the three primaries. The chromaticity coordinates r , g , and b specify color without specifying luminance. They are defined in terms of the tristimulus values as follows:

$$\begin{aligned} r &= R / (R + G + B), \\ g &= G / (R + G + B), \\ b &= B / (R + G + B). \end{aligned}$$

Since $r + g + b = 1$ (why?), only two chromaticity coordinates are independent. Like the tristimulus values, the chromaticity coordinates may be

represented as a vector (r,g,b) . The chromaticity coordinates of white are $(1/3, 1/3, 1/3)$. The chromaticity coordinates of our pink example are $(1/2, 1/4, 1/4)$. The chromaticity coordinates of a 700 nm light of *any* luminance would be $(1, 0, 0)$.

Two assumptions underly all of this: (1) hue and saturation, the color information about a light source, are independent of the luminance of the light source; (2) light sources obey Abney's law, that is lumens add algebraically regardless of the colors of the contributing light sources. Though there has been argument about the validity of these assumptions, we will continue to accept them here.

THE CIE COLORIMETRIC SYSTEM

The results of color matching experiments done with one set of primaries may be converted to those done with another set of primaries by using a linear transformation, a so-called linear transform which represents the chromaticity vector in a new coordinate system. In this way, for example, David Wright was able to compare his distribution coefficients taken with 650 nm, 530 nm, and 460 nm primaries with J. Guild's data taken using 700 nm, 546 nm, and 436 nm primaries.

In 1931 the Comission Internationale d'Eclairage (CIE) adopted a standard system for color specification. The CIE, in its wisdom, chose to use imaginary primaries designated X, Y, Z. (Here's where those darned imaginary colors come in.)

The tristimulus values in the CIE system are (X, Y, Z) . The chromaticity coordinates are (x, y, z) , and they are related to the tristimulus values as in the rgb system:

$$\begin{aligned}x &= X/(X+Y+Z), \\y &= Y/(X+Y+Z), \\z &= Z/(X+Y+Z).\end{aligned}$$

The distribution coefficients are x_λ , y_λ , and z_λ . These are plotted in figure 4 and tabulated in table 1.

¹Distribution coefficients are more commonly notated as a variable over which there is a bar.

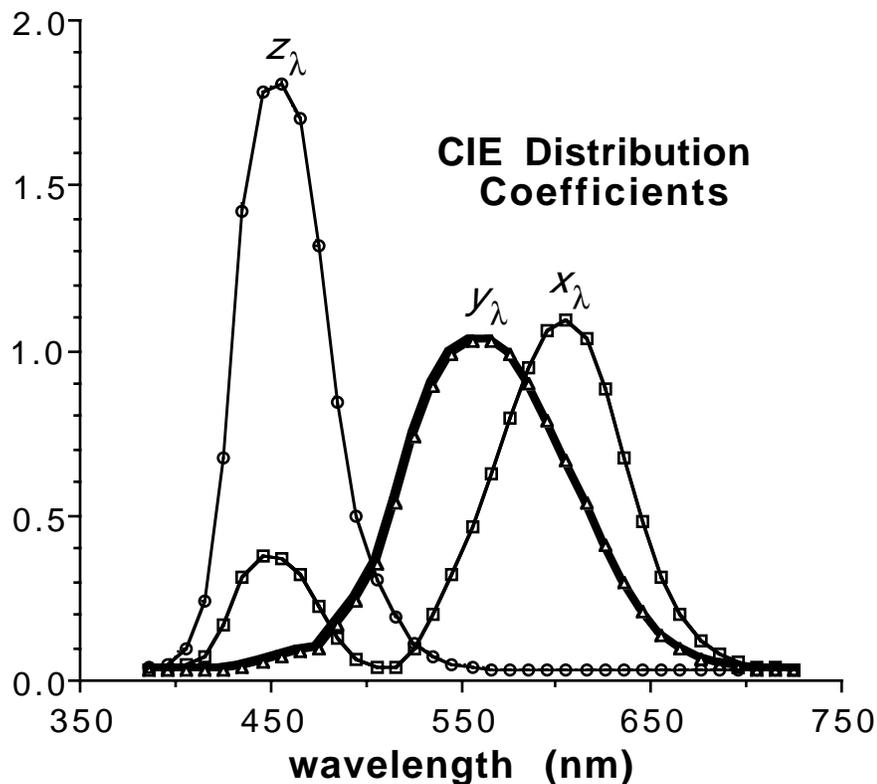


Figure 4. Distribution coefficients in the CIE system.

Note the familiar shape of the y_λ curve. It is, in fact, an old friend the spectral sensitivity function v_λ .

Making the distribution function of the Y color the same as the spectral sensitivity function is only one of the mathematical miracles of the CIE system. Another is that the distribution functions are all positive. The chromaticity coordinates of all real light sources are positive. And the system *still* allows white light to have chromaticity coordinates (1/3, 1/3, 1/3). All that has been sacrificed in all this mathematical elegance is any physical intuition. While many think the CIE system a clever mess, it's standard and we're stuck with it!

The chromaticity coordinates of the spectral colors may be calculated from the distribution functions remembering that the distribution functions are just tristimulus values. So for a spectral color of wavelength λ , the color coordinates are

$$\begin{aligned} x &= x_\lambda / (x_\lambda + y_\lambda + z_\lambda), \\ y &= y_\lambda / (x_\lambda + y_\lambda + z_\lambda), \\ z &= z_\lambda / (x_\lambda + y_\lambda + z_\lambda). \end{aligned}$$

λ	x_λ	y_λ	z_λ
380	0.001	0.000	0.006
390	0.004	0.000	0.020
400	0.014	0.000	0.068
410	0.044	0.001	0.207
420	0.134	0.004	0.646
430	0.283	0.012	1.386
440	0.348	0.023	1.747
450	0.336	0.038	1.772
460	0.291	0.060	1.669
470	0.195	0.061	1.288
480	0.096	0.139	0.813
490	0.032	0.208	0.465
500	0.005	0.323	0.272
510	0.009	0.503	0.158
520	0.063	0.710	0.078
530	0.166	0.862	0.042
540	0.290	0.954	0.020
550	0.433	0.995	0.009
560	0.594	0.995	0.004
570	0.762	0.952	0.002
580	0.916	0.870	0.002
590	1.026	0.757	0.001
600	1.062	0.631	0.001
610	1.002	0.503	0.000
620	0.854	0.381	0.000
630	0.642	0.265	0.000
640	0.448	0.175	0.000
650	0.284	0.107	0.000
660	0.165	0.061	0.000
670	0.087	0.032	0.000
680	0.047	0.017	0.000
690	0.023	0.008	0.000
700	0.011	0.004	0.000
710	0.006	0.002	0.000
720	0.003	0.001	0.000

Table 1. Distribution coefficients in the CIE system given to three places.

For 550 nm spectral light, for example, the chromaticity coordinates are

$$\begin{aligned}
 x &= 0.433 / (0.433 + 0.995 + 0.009) = 0.30, \\
 y &= 0.995 / (0.433 + 0.995 + 0.009) = 0.69, \\
 z &= 0.009 / (0.433 + 0.995 + 0.009) = 0.01.
 \end{aligned}$$

These, of course, sum to one, as they must.

THE CHROMATICITY DIAGRAM

It is useful in visualizing the CIE system and in doing certain practical calculations to plot the chromaticity coordinates of the spectral colors on a diagram. Since only two of the chromaticity coordinates are independent, we plot only the x and y chromaticity coordinates to get the diagram of figure 5.

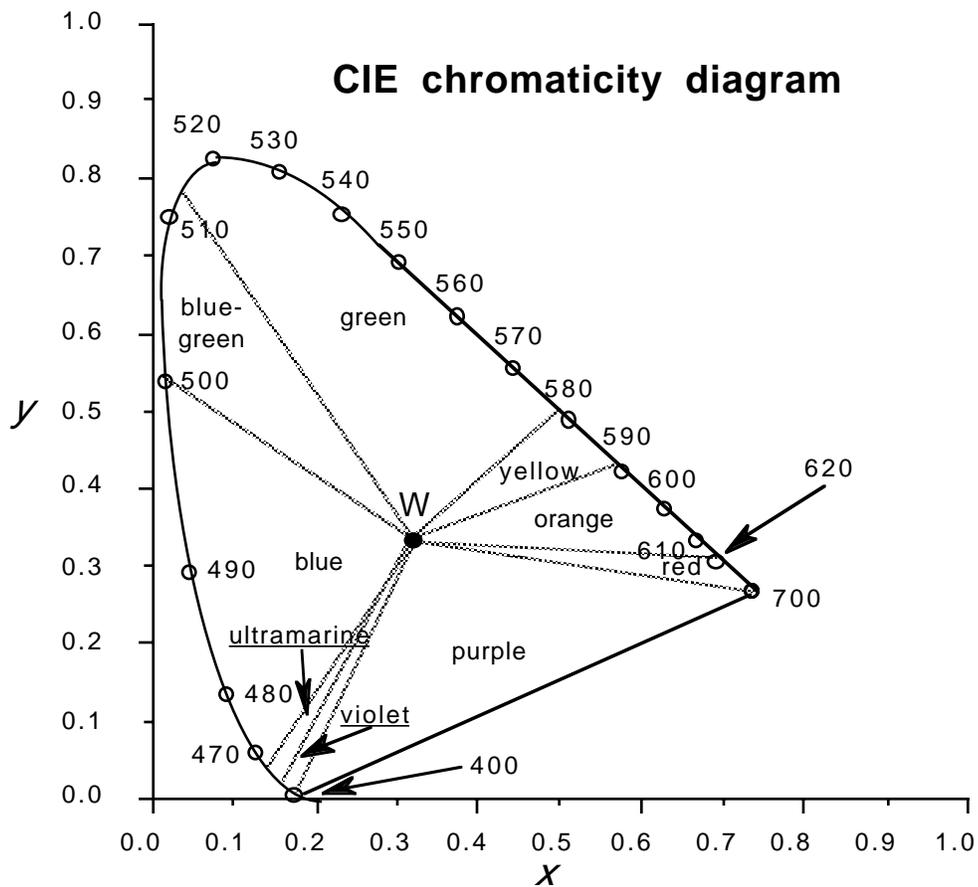


Figure 5. The CIE chromaticity diagram.

The spectral colors lie on a curved, convex locus. In the chromaticity diagram, the wavelength (in nm) of each spectral source is written next to the circle indicating its chromaticity coordinates. The usual color names associated with wavelengths are in table 2. The shorter wavelengths are widely spaced on the convex portion of the locus to the left. Longer wavelengths are closer together along an almost straight locus to the right. The points corresponding to wavelengths at the extreme ends of the visual spectrum are crowded together. The point corresponding to white light with chromaticity coordinates $(1/3, 1/3)$ is designated W.

<u>wavelength range</u>	<u>color name</u>
greater than 620 nm	red
620 nm to 592 nm	orange
592 nm to 578 nm	yellow
578 nm to 513 nm	green
513 nm to 500 nm	blue-green
500 nm to 464 nm	blue
464 nm to 446 nm	ultramarine
less than 446 nm	violet

Table 2. Color names corresponding to wavelengths.

A useful feature of this sort of representation is that the chromaticity coordinates (x, y) of a color achieved by combining two light sources represented by chromaticity coordinates (x_1, y_1) and (x_2, y_2) lies along the line connecting (x_1, y_1) and (x_2, y_2) on the chromaticity diagram, as shown in figure 6.

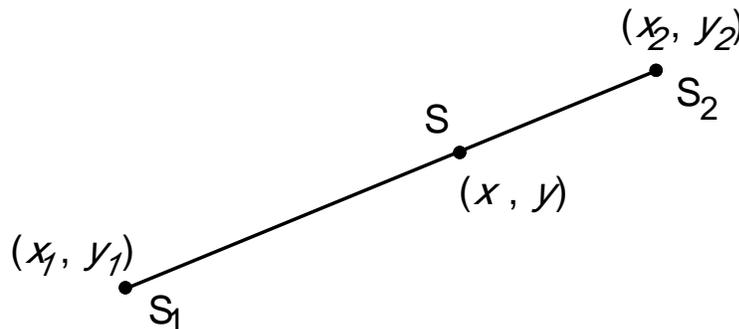


Figure 6. When two colors S_1 and S_2 are mixed, their resultant S lies at the center of gravity between their points on the chromaticity diagram.

If S_1 has luminance L_1 and S_2 has luminance L_2 , then

$$S_1S/S_2S=L_1/L_2.$$

Thus the chromaticity of the combination lies closest to the chromaticity coordinate of the stronger source. If both sources have equal luminance, the combination lies halfway between. If one source has zero luminance, the chromaticity coordinate of the combination coincides with the other source. Thus the combination lies at the center of gravity of the two sources.

The colors achieved by combining colors at short and long wavelength extremes of the visible spectrum are on the straight line portion of the spectral locus of the CIE diagram. These colors are purples.

We can achieve the same color an infinite number of ways. In figure 7, for example, the color represented by point S can be obtained by combining S_1 and S_2 or S_3 and S_4 in the proper proportions. Both combinations will look the same to an observer, though the spectral composition of their sources are quite different. These are called metamers.

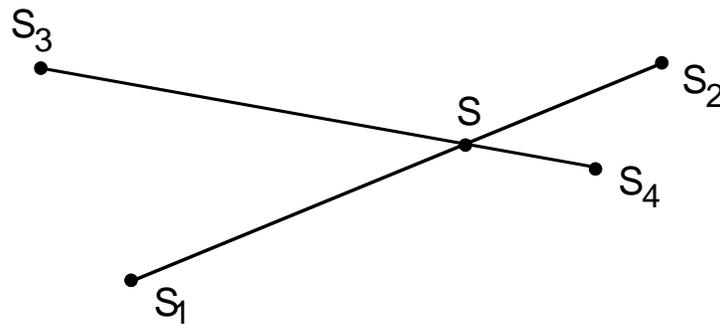


Figure 7. The color represented by point S may be achieved by combining sources S_1 and S_2 or S_3 and S_4 in the proper proportions. The two combinations are metamers which appear the same to observers.

If we add white light to a spectral source, the combination lies along the line connecting the chromaticity of the spectral source along the spectral locus and the point W. The closer the point to W the more desaturated the color. If λ is the point on the spectral locus and S a source point somewhere within the spectral locus, the excitation purity, ρ_e , of the source is

$$\rho_e = WS/W\lambda.$$

The dominant hue of a source S may be determined from the intersection of the line through W and S with the spectral locus. In figure 5, the dominant color names associated with dominant hues are indicated in wedges centered at point W. Any color with chromaticity coordinates within one of these wedges has the indicated color name, the points closer to W being more desaturated.

For every color there is a complementary color which can be combined with it in suitable proportions to produce white. Clearly the line connecting complementary colors must intersect W (figure 8).

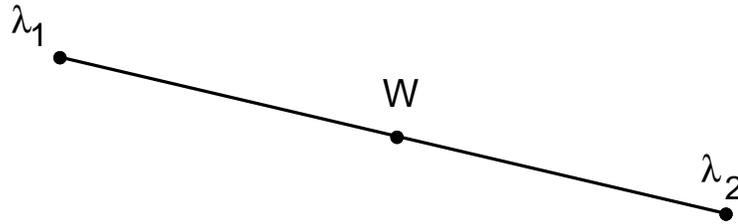


Figure 8. The line connecting complementary colors λ_1 and λ_2 passes through W, the point corresponding to white light.

CALCULATIONS IN THE CIE SYSTEM

We can use the theory presented above to find the color, chromaticity coordinates, saturation, and complements of a light source or combination of light sources. In this section we'll give some examples.

Example 1. Find the chromaticity coordinates and color of an "equal energy" source, a source for which $P_\lambda = \text{const}$.

Solution: Since the chromaticity coordinates are relative values independent of the radiance of a source, we can, for simplicity let $P_\lambda = 1$. Recalling that the distribution functions are the tristimulus values of unit energy sources, the tristimulus values of the combination of sources is

$$\begin{aligned} X &= \sum(x_\lambda P_\lambda), \\ Y &= \sum(y_\lambda P_\lambda), \\ Z &= \sum(z_\lambda P_\lambda). \end{aligned}$$

(Technically we should integrate over the energy distribution, but in practice we have to use Simpson's rule which gives the sums above.) The chromaticity coordinates can be calculated from the tristimulus coefficients. We can lay out our calculation on a spreadsheet as shown below:

λ	x_λ	y_λ	z_λ	P_λ	$x_\lambda P_\lambda$	$y_\lambda P_\lambda$	$z_\lambda P_\lambda$
380	0.001	0.000	0.006	1.000	0.001	0.000	0.006
390	0.004	0.000	0.020	1.000	0.004	0.000	0.020
400	0.014	0.000	0.068	1.000	0.014	0.000	0.068
410	0.044	0.001	0.207	1.000	0.044	0.001	0.207
420	0.134	0.004	0.646	1.000	0.134	0.004	0.646
430	0.283	0.012	1.386	1.000	0.283	0.012	1.386
440	0.348	0.023	1.747	1.000	0.348	0.023	1.747
450	0.336	0.038	1.772	1.000	0.336	0.038	1.772
460	0.291	0.060	1.669	1.000	0.291	0.060	1.669
470	0.195	0.061	1.288	1.000	0.195	0.061	1.288
480	0.096	0.139	0.813	1.000	0.096	0.139	0.813
490	0.032	0.208	0.465	1.000	0.032	0.208	0.465
500	0.005	0.323	0.272	1.000	0.005	0.323	0.272
510	0.009	0.503	0.158	1.000	0.009	0.503	0.158
520	0.063	0.710	0.078	1.000	0.063	0.710	0.078
530	0.166	0.862	0.042	1.000	0.166	0.862	0.042
540	0.290	0.954	0.020	1.000	0.290	0.954	0.020
550	0.433	0.995	0.009	1.000	0.433	0.995	0.009
560	0.594	0.995	0.004	1.000	0.594	0.995	0.004
570	0.762	0.952	0.002	1.000	0.762	0.952	0.002
580	0.916	0.870	0.002	1.000	0.916	0.870	0.002
590	1.026	0.757	0.001	1.000	1.026	0.757	0.001
600	1.062	0.631	0.001	1.000	1.062	0.631	0.001
610	1.002	0.503	0.000	1.000	1.002	0.503	0.000
620	0.854	0.381	0.000	1.000	0.854	0.381	0.000
630	0.642	0.265	0.000	1.000	0.642	0.265	0.000
640	0.448	0.175	0.000	1.000	0.448	0.175	0.000
650	0.284	0.107	0.000	1.000	0.284	0.107	0.000
660	0.165	0.061	0.000	1.000	0.165	0.061	0.000
670	0.087	0.032	0.000	1.000	0.087	0.032	0.000
680	0.047	0.017	0.000	1.000	0.047	0.017	0.000
690	0.023	0.008	0.000	1.000	0.023	0.008	0.000
700	0.011	0.004	0.000	1.000	0.011	0.004	0.000
710	0.006	0.002	0.000	1.000	0.006	0.002	0.000
720	0.003	0.001	0.000	1.000	0.003	0.001	0.000
					$X=\Sigma(x_\lambda P_\lambda)$	$Y=\Sigma(y_\lambda P_\lambda)$	$Z=\Sigma(z_\lambda P_\lambda)$
					10.676	10.654	10.676
					$x=X/(X+Y+Z)$	$y=Y/(X+Y+Z)$	$z=Z/(X+Y+Z)$
					0.33	0.33	0.33

Thus the chromaticity coordinates of an equal energy source are (0.33,0.33,0.33). This, however, is point W, the location of a white source, so the equal energy source is white.

Example 2. Find the chromaticity coordinates, color, saturation, and complementary color for white light from an equal energy source transmitted through a filter with spectral transmittance shown in the graph below.

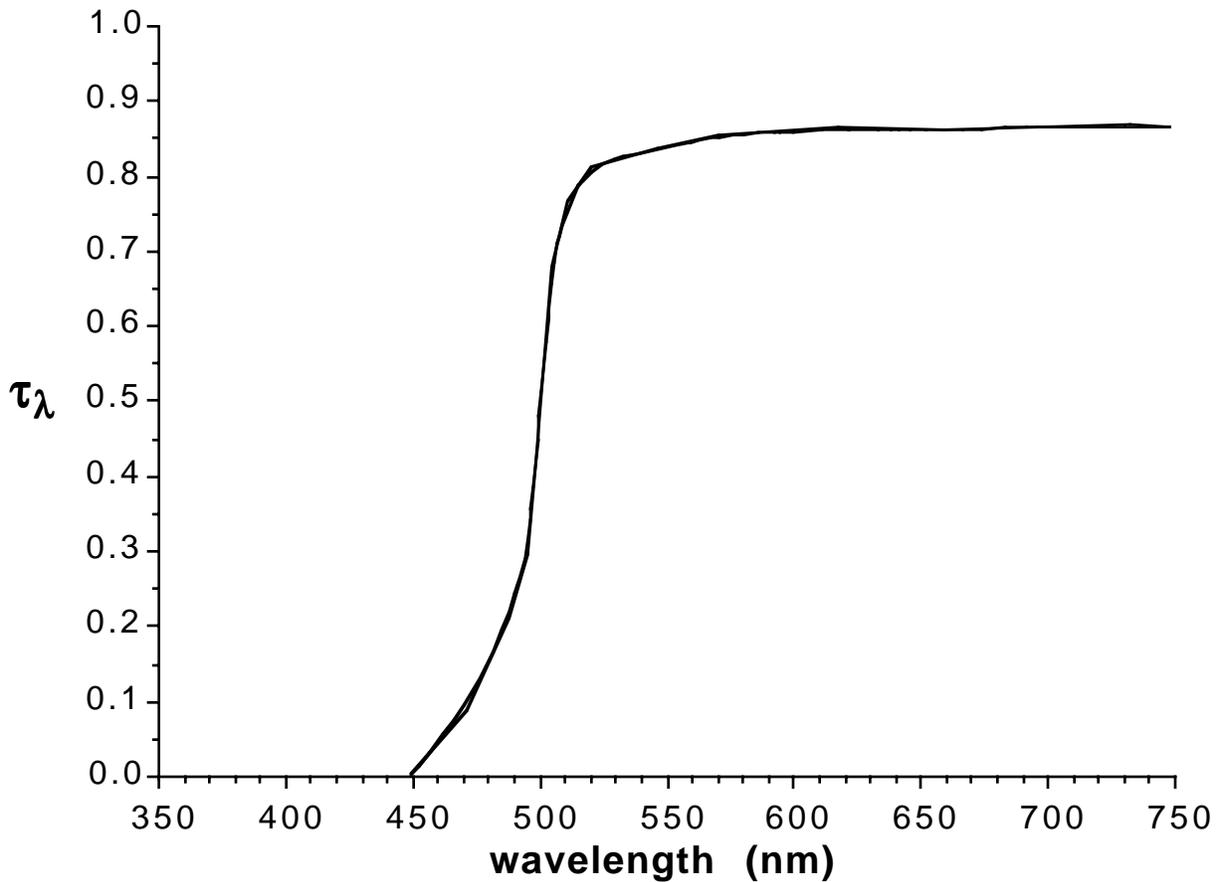


Figure 9. The spectral transmittance of a Kalichrome lens.

Solution: For light from an equal energy light source going through a filter with spectral transmittant τ_λ $P_\lambda \propto \tau_\lambda$

$$\begin{aligned}
 X &= \Sigma(x_\lambda \tau_\lambda), \\
 Y &= \Sigma(y_\lambda \tau_\lambda), \\
 Z &= \Sigma(z_\lambda \tau_\lambda).
 \end{aligned}$$

We can calculate tristimulus values from the distribution coefficients as we did in example 1, using the spreadsheet below. The values of spectral transmittance are read from the graph above.

λ	x_λ	y_λ	z_λ	τ_λ	$x_\lambda \tau_\lambda$	$y_\lambda \tau_\lambda$	$z_\lambda \tau_\lambda$
380	0.001	0.000	0.006	0.00	0.00	0.00	0.00
390	0.004	0.000	0.020	0.00	0.00	0.00	0.00
400	0.014	0.000	0.068	0.00	0.00	0.00	0.00
410	0.044	0.001	0.207	0.00	0.00	0.00	0.00
420	0.134	0.004	0.646	0.00	0.00	0.00	0.00
430	0.283	0.012	1.386	0.00	0.00	0.00	0.00
440	0.348	0.023	1.747	0.00	0.00	0.00	0.00
450	0.336	0.038	1.772	0.00	0.00	0.00	0.00
460	0.291	0.060	1.669	0.03	0.01	0.00	0.05
470	0.195	0.061	1.288	0.06	0.01	0.00	0.08
480	0.096	0.139	0.813	0.12	0.01	0.02	0.10
490	0.032	0.208	0.465	0.18	0.01	0.04	0.08
500	0.005	0.323	0.272	0.27	0.00	0.09	0.07
510	0.009	0.503	0.158	0.38	0.00	0.19	0.06
520	0.063	0.710	0.078	0.81	0.05	0.58	0.06
530	0.166	0.862	0.042	0.82	0.14	0.71	0.03
540	0.290	0.954	0.020	0.83	0.24	0.79	0.02
550	0.433	0.995	0.009	0.84	0.36	0.84	0.01
560	0.594	0.995	0.004	0.85	0.50	0.85	0.00
570	0.762	0.952	0.002	0.85	0.65	0.81	0.00
580	0.916	0.870	0.002	0.85	0.78	0.74	0.00
590	1.026	0.757	0.001	0.85	0.87	0.64	0.00
600	1.062	0.631	0.001	0.85	0.90	0.54	0.00
610	1.002	0.503	0.000	0.85	0.85	0.43	0.00
620	0.854	0.381	0.000	0.85	0.73	0.32	0.00
630	0.642	0.265	0.000	0.85	0.55	0.23	0.00
640	0.448	0.175	0.000	0.85	0.38	0.15	0.00
650	0.284	0.107	0.000	0.85	0.24	0.09	0.00
660	0.165	0.061	0.000	0.85	0.14	0.05	0.00
670	0.087	0.032	0.000	0.85	0.07	0.03	0.00
680	0.047	0.017	0.000	0.85	0.04	0.01	0.00
690	0.023	0.008	0.000	0.85	0.02	0.01	0.00
700	0.011	0.004	0.000	0.85	0.01	0.00	0.00
710	0.006	0.002	0.000	0.85	0.01	0.00	0.00
720	0.003	0.001	0.000	0.85	0.00	0.00	0.00
					$X=\Sigma(x_\lambda \tau_\lambda)$	$Y=\Sigma(y_\lambda \tau_\lambda)$	$Z=\Sigma(z_\lambda \tau_\lambda)$
					7.58	8.14	0.57
					$x=X/(X+Y+Z)$	$y=Y/(X+Y+Z)$	$z=Z/(X+Y+Z)$
					0.47	0.50	0.04

So the chromaticity coordinates are, finally, (0.47,0.50,0.04). The x and y coordinates on the CIE chromaticity diagram are at the gray box in the figure below, labelled S. From its position on the diagram, it is clear that the color of the filter is a quite saturated green–yellow. This figures since the spectral transmittance curve is that of a Kalichrome filter, the yellow lens material used in the classic "shooter's glasses" favored by sportsmen.

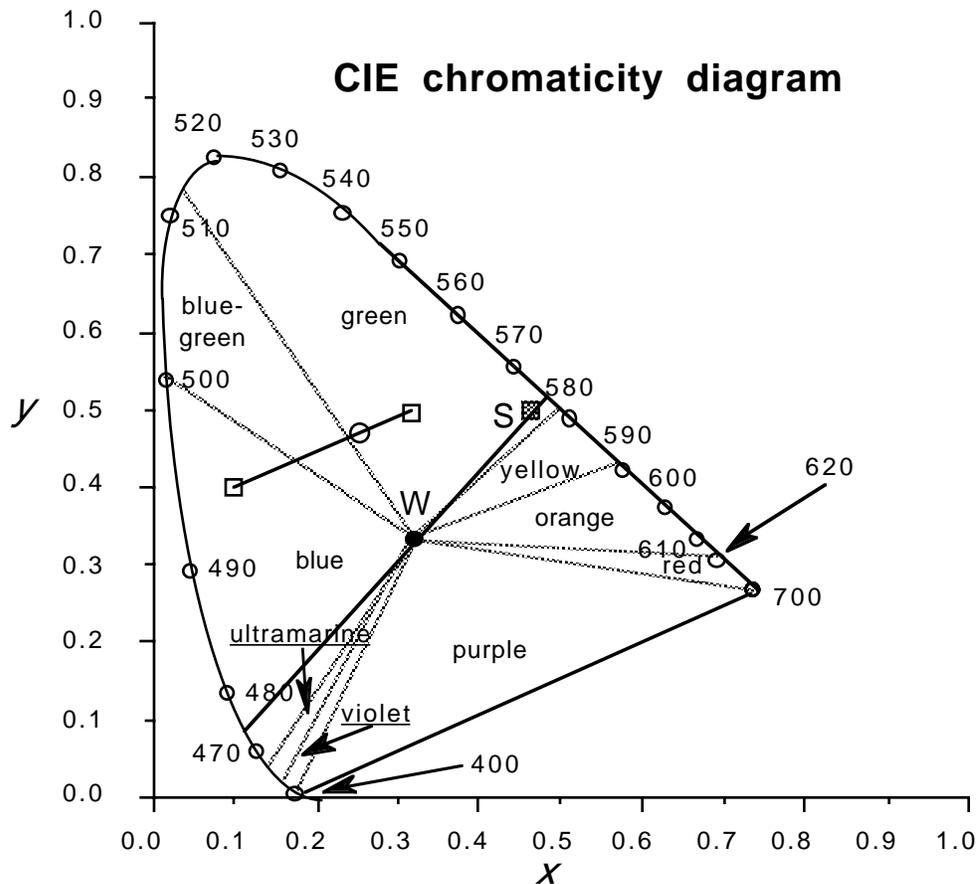


Figure 10. The gray square is at the chromaticity coordinates of the Kalichrome lens of example 2. The open squares are at the chromaticity of the two stimuli of example 3 with the open circle at the chromaticity coordinates of the combination of these two stimuli.

Drawing a line from WS and extending it to the spectral locus, we see the dominant hue is 576 nm. Extending the line to the opposite side of the spectral locus, we complementary color is 472 nm blue. Taking the ratio of the line segment WS to the distance from W to the spectral locus gives the excitation purity, about 90%.

Example 3. An observer sees a stimulus composed of 20 ft-L of blue light with chromaticity coordinates (0.10, 0.40, 0.50) and 60 ft-L of green light with chromaticity coordinates (0.30, 0.50, 0.20). What are the luminance, color coordinates, and dominant hue of the stimulus?

Solution: By Abney's law, the luminance is simply 80 ft-L. We can obtain the color coordinates of the combination graphically or algebraically. Algebraically note that the tristimulus value of each color component is its luminance times its chromaticity vector. Summing these tristimulus values to get the tristimulus vector of the stimulus

$$(X, Y, Z) = 20(0.10, 0.40, 0.50) + 60(0.30, 0.50, 0.20) = (20, 38, 22).$$

The chromaticity coordinates of the combined stimuli is then

$$(x, y, z) = [1/(20+38+22)](20, 38, 22) = (0.25, 0.47, 0.27).$$

The chromaticity coordinates of the two stimuli are plotted with hollow squares on the diagram above. The resultant is plotted as a hollow circle. As predicted by the center of gravity rule, the circle lies on the line connecting the two stimuli, one fourth the way from the stronger of the two stimuli (why?). From its location on the CIE diagram, the resultant is green, nearly blue-green.

ADDITIVE AND SUBTRACTIVE COLOR SYSTEMS

The discussion above dealt with additive color systems in which lights are superimposed. Many practical systems deal with subtractive color systems, i.e. the mixing of paints or other tints. In an additive system, combined complementary colors produce white light. In a subtractive system they produce black. With appropriate adjustments, the colorimetry systems above may be applied to either kind of system.

CONCLUSION

Colorimetry is essentially a **descriptive** system. It summarizes the results of certain color matching experiments and allows us to apply those results to a number of real-life situations. But colorimetry doesn't **explain** the sensation of color or how the eye and brain process color information. It merely provides a means of quantifying color information.